HOW TO WORK WITH CURVED STRUCTURES; THEORY

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EXTENDED ABSTRACT

Teaching of structural mechanics – or mechanics generally - is traditionally an extremely challenging task. Curved structures particularly have proven to be frightening for the students. For “straight” structural members the basis unit vectors applied can be kept as constants. Usually simple figures describing a material element in the original and in the deformed state produce rather straightforwardly the required strain expressions. Similarly, simple free body diagrams for the material element are enough to give the local equilibrium equations and the traction boundary conditions.

Curved structures need to be described initially in curvilinear coordinates. The unit basis vectors are then no more constants but depend on the position. This is one major complication as compared to the straight case. Perhaps a still more serious difficulty concerns the use of figures and diagrams to produce the strains and equilibrium equations. The figures and diagrams tend to become so involved that a doubt about the correctness of the deductions can easily emerge in the mind of the student. A general unifying background theory is easily lost as more or less ad hoc figures are employed in each new structural case. Further, if large deformation problems are considered, correct deductions in this way are in practice out of the question. Of course, tensor calculus in curvilinear coordinates solves these problems elegantly, but this approach is quite too demanding in time and effort to be used in basic structural mechanics courses.

An approach which produces the relevant expression in curvilinear orthogonal coordinates without tensor calculus is described in the paper. The approach is called “the method of local Cartesian frame”. The main idea is: if we have an expression valid in rectangular Cartesian coordinates, a corresponding expression for curvilinear orthogonal coordinates can be formed in simple steps. Explaining the steps to the students does not demand too much effort. A general theory is described in the paper. For instance gradient and divergence expression and general small deformation strain expressions in two dimensions are developed. No figures describing the material element in the initial and deformed state are used. Again, such figures are fine in simple straight structure applications but are, in our opinion, not convincing enough in curved cases.

The equilibrium equations for curved structures are not derived here directly by the method of local Cartesian frame although this can be done. Alternatively, after the strains have been arrived at, the principle of virtual work is used for this purpose. The importance of the principle of virtual work in structural mechanics cannot be overemphasized. It unifies analytical and numerical approaches - especially the finite element method. Here with curved structures it is employed to derive the local equilibrium equations and traction boundary conditions. The invariance properties of the internal and external virtual work are employed when using the method of a local Cartesian frame. Integration by parts operations are needed in the manipulations. No free body diagrams are applied.

KEYWORDS
Teaching Techniques for Mechanics, Curved Structures, The Method of Local Cartesian Frame, Principle of Virtual Work.

1. INTRODUCTION

Education processes are – nowadays particularly – under strong development. The challenges are obvious because the interest towards theoretical studies has decreased rapidly. One reason for this is the highly developed numerical technique and computers with versatile possibilities in structural analysis. Though the profits achieved, its almightiness gives a signal that theoretical capability belongs nowadays to the computers only. This will ruin the interest to study theoretical subjects destroying at the same time the general development of mathematical thinking and manual problem solving as well.

This has forced teachers to think also the methodology applied in teaching. Structural analysis has been too method oriented so far. University studies have included tens of different strategies to solve a single problem. In addition, all the areas of mechanics have been far too split, and no common 'red line' for the analysis in general does exist.

There is clearly a lack of common mathematical tools for structural analyses. These should be mathematically exact supported on very basic mathematics only, and they should be usable in all areas of mechanics. When getting familiar with these tools in basic courses already, a student does not loose his interest when the problems become more complicated. Applying the differential geometry and its figures is graphic and works well with simple structures, but when the structures considered are more complicated or in non-linear analyses they will lose their usability.

Structural mechanics is actually a strong combination of mechanics and mathematics. The problem is how to succeed to put more emphasis on the part of mechanics and push the mathematics – which has nowadays a very limited popularity in various university syllabuses – more to the background. Solving different complicated partial or ordinary differential equations belongs to the pure mathematics though it often takes the biggest part of the attention in various courses of mechanics. The procedures looked for are concentrating on the mechanics, and the mathematics is aimed to play the side role only. The goal is to find out a mathematical tool which could be applied to any structural problem in any geometry – including curvilinear structures. In this way, we will have a lot more time in getting familiar with the problems of pure mechanics.

We are proposing in this paper a combination of methods to be applied when building up a model for a generic structure. The procedure consists of a method of local Cartesian frame and the principle of virtual work. These form up together a simple logical way to handle any type of structural task.

2. MATHEMATICAL BACKGROUND AND GEOMETRY DESCRIPTION

In this chapter, some basic formulas dealing with orthogonal curvilinear coordinates are reviewed. The presentation is given in two dimensions as the main points can be seen already in this case.

The starting point when looking for new ways to approach a generic structural problem is, at first, to realize the fact that the geometry and also the deformation are actually vector quantities, and try to find suitable ways to exploit it. The geometry of any structure can always be fixed by a position vector $\mathbf{r}$ the origin of which is placed at the origin of any global Cartesian coordinate system x, y. The curvilinear geometries are defined by local curvilinear coordinates, α, β, coinciding with the geometry of the body, or characteristic lines of the structure considered.

The meaning of this vector is great. It includes a lot of important information and defines the space where the kinematics and boundary conditions are most naturally given. The fact, that this particular vector includes in addition all the information needed to define any differential operator in the differential equation to be solved, is a huge profit. And these operators are defined in any arbitrary coordinate curvilinear system α, β, which has not been emphasized too much usually anywhere.

A rectangular Cartesian coordinate system x, y with unit base vectors $\mathbf{i}, \mathbf{j}$ and a curvilinear orthogonal system α, β with unit vectors $\mathbf{e}_\alpha, \mathbf{e}_\beta$ are considered (Figure 1).

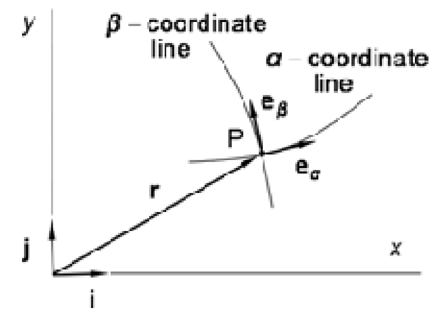

Figure 1: Two coordinate systems.

The coordinates fixing the position vector are connected by

$$x = x(\alpha,\beta), \; y = y(\alpha,\beta). \tag{1}$$

The position vector **r** can be expressed as

$$\mathbf{r} = \mathbf{r}(\alpha,\beta) = x(\alpha,\beta)\mathbf{i} + y(\alpha,\beta)\mathbf{j} \tag{2}$$

or alternatively as

$$\mathbf{r} = \mathbf{r}(\alpha,\beta) = r_\alpha(\alpha,\beta)\mathbf{e}_\alpha(\alpha,\beta) + r_\beta(\alpha,\beta)\mathbf{e}_\beta(\alpha,\beta). \tag{3}$$

with

$$\mathbf{r}_\alpha(\alpha,\beta) = \mathbf{r}(\alpha,\beta)\cdot\mathbf{e}_\alpha(\alpha,\beta), \quad \mathbf{r}_\beta(\alpha,\beta) = \mathbf{r}(\alpha,\beta)\cdot\mathbf{e}_\beta(\alpha,\beta) \tag{4}$$

The partial derivatives $\partial\mathbf{r}/\partial\alpha$ and $\partial\mathbf{r}/\partial\beta$ of the position vector **r** with respect to the curvilinear coordinates are tangent vectors to the corresponding coordinate lines and one can thus write

$$\frac{\partial\mathbf{r}}{\partial\alpha} = h_\alpha\mathbf{e}_\alpha\,, \;\; \frac{\partial\mathbf{r}}{\partial\beta} = h_\beta\mathbf{e}_\beta\,, \tag{5}$$

where the scale factors $h_\alpha = |\partial\mathbf{r}/\partial\alpha|$, $h_\beta = |\partial\mathbf{r}/\partial\beta|$ are obtained from

$$h_\alpha = \left[\left(\frac{\partial x}{\partial\alpha}\right)^2 + \left(\frac{\partial y}{\partial\alpha}\right)^2\right]^{1/2}, \quad h_\beta = \left[\left(\frac{\partial x}{\partial\beta}\right)^2 + \left(\frac{\partial y}{\partial\beta}\right)^2\right]^{1/2}. \tag{6}$$

These are arrived at by differentiating expression (2) where the Cartesian unit vectors are constants. The derivatives of the unit vectors are also needed. There is obtained

$$\frac{\partial \mathbf{e}_\alpha}{\partial \alpha} = -\frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\mathbf{e}_\beta , \quad \frac{\partial \mathbf{e}_\alpha}{\partial \beta} = \frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\mathbf{e}_\beta ,$$
$$\frac{\partial \mathbf{e}_\beta}{\partial \alpha} = \frac{1}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\mathbf{e}_\alpha , \quad \frac{\partial \mathbf{e}_\beta}{\partial \beta} = -\frac{1}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\mathbf{e}_\alpha . \tag{7}$$

These are found by manipulations of the equation arrived at from further differentiation of the first and second equation (5) with respect to β and α, respectively.

The polar coordinates $\alpha \triangleq r$, $\beta \triangleq \theta$ (Figure 2) are employed here and in the following as a simple specific illustrative example case.

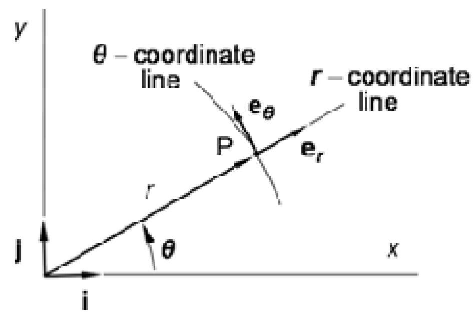

Figure 2: Polar coordinate system.

From Figure 2,

$$x = r\,cos\theta , \quad y = r\,sin\theta . \tag{8}$$

Corresponding to (2) and (3), we have

$$\mathbf{r}(r,\theta) = r\,cos\theta\,\mathbf{i} + r\,sin\theta\,\mathbf{j} \tag{9}$$

and

$$\mathbf{r}(r,\theta) = r\mathbf{e}_r(\theta) + 0\cdot\mathbf{e}_\theta(\theta). \tag{10}$$

The scale factors (6) are

$$h_r = \left[cos^2\theta + sin^2\theta\right]^{1/2} = 1, \quad h_\theta = \left[r^2 sin^2\theta + r^2 cos^2\theta\right]^{1/2} = r \tag{11}$$

and the derivatives (6) are

$$\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0}, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta , \quad \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}, \quad \frac{\partial \mathrm{e}_\theta}{\partial \theta} = -\mathbf{e}_r . \tag{12}$$

In this simple application, the derivatives (12) can also be found directly by inspection without the use of the general expressions (7).

3. HOW TO CALCULATE DERIVATIVES IN CURVILINEAR COORDINATES

Any problem of mechanics ends up to various ordinary or partial differential equations. Therefore, the concept of differentiation must be well-understood, particularly when considering differentiation in curvilinear coordinates which has proven to be rather difficult to handle traditionally. A derivative or gradient measures the change of the function considered. To have a fixed frame with respect to which the measurement will be done, we can apply a local Cartesian frame, and orientate its unit base vectors to coincide with the unit vectors of the curvilinear system at the point where the results will be reached. A curvilinear coordinate do not form a proper system to work as such because the direction of its unit vectors are changing from point to point. The idea is to perform the mathematical operations in these locally defined orthogonal rectilinear coordinates where all the well known rules are valid, and consider finally the result at the single point where the coordinates are equal. In this way, we can bypass all the operations in curvilinear coordinates.

An auxiliary local Cartesian coordinate system X, Y — or shortly a local Cartesian frame — with unit base vectors $\mathbf{e}_X$, $\mathbf{e}_Y$ is made use of with its origin at a generic point P and its axes tangent to the α-, β-coordinate lines (Figure 3). This local frame can be brought to any point P but it is very important to stress to the students that *during a specific derivation of a result, the frame is considered fixed* so that the unit vectors $\mathbf{e}_X$ and $\mathbf{e}_Y$ are constants with respect to differentiation.

At the local origin — and not elsewhere in general —

$$\mathbf{e}_X = \mathbf{e}_\alpha\,, \quad \mathbf{e}_Y = \mathbf{e}_\beta \tag{13}$$

When applying this method, a rule between differentiation in the local Cartesian and curvilinear coordinates will be needed. This is easy to build up by applying the well known chain rule which will be simplified in orthogonal systems to a diagonal form to give

$$\frac{\partial}{\partial X} = \frac{1}{h_\alpha}\frac{\partial}{\partial \alpha} \quad , \quad \frac{\partial}{\partial Y} = \frac{1}{h_\beta}\frac{\partial}{\partial \beta}\ . \tag{14}$$

These are the formulas which are employed repeatedly in the applications to follow. Relations (13) should be obvious from Figure 3. Further, the first formula (14), for instance, can be clarified as follows. From (5) due to an increment $\mathrm{d}\alpha$, the position vector obtains the increment $\mathrm{d}\mathbf{r} = h_\alpha \mathbf{e}_\alpha \mathrm{d}\alpha = h_\alpha \mathrm{d}\alpha \mathbf{e}_\alpha$ which equals $\mathrm{d}X\mathbf{e}_X = \mathrm{d}X\mathbf{e}_\alpha$ so $\mathrm{d}X = h_\alpha \mathrm{d}\alpha$.

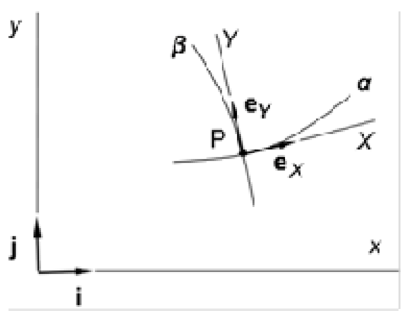

Figure 3: The local frame.

The local frame can be considered as a tool which is taken temporarily in use and then discarded as its function has been fulfilled. We will call this kind of application of the local frame as *the method of local Cartesian frame*.

2.2 Displacement field

Another important vector is the displacement vector defining the kinematics chosen for the structure considered. This theme, the role of which is strongly undervalued in teaching, is the kinematics. It is an extremely powerful tool. Usually, it will not be explained explicitly, that the difference between various beam, plate and shell theories is hiding in the kinematics adopted. The kinematics is the way to define the whole problem to be solved. It is also worth noticing that it is a tool controlled by the analyzer himself. The geometry, loading, the accuracy looked for, are the factors to be taken into account, when choosing proper kinematics,

The displacement field ***u*** has the alternative representations

$$\begin{aligned}\mathbf{u} &= u\mathbf{i} + v\mathbf{j}, \\ &= u_X\mathbf{e}_X + u_Y\mathbf{e}_Y, \\ &= u_\alpha\mathbf{e}_\alpha + u_\beta\mathbf{e}_\beta .\end{aligned} \tag{15}$$

Where, for example

$$u = \mathbf{u}\cdot\mathbf{i},\ v = \mathbf{u}\cdot\mathbf{j},\text{ and } u_X = \mathbf{u}\cdot\mathbf{e}_X,\ u_Y = \mathbf{u}\cdot\mathbf{e}_Y \tag{16}$$

The well-known expressions for small strain components in Cartesian coordinates are

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x}, \\ \varepsilon_y &= \frac{\partial v}{\partial y}, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.\end{aligned} \tag{17}$$

It is very useful to write the definitions by applying the vector presentation of the displacement field by replacing the components (16) into the definitions (17), which yields

$$\begin{aligned}\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{i}, \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{j}, \\ \gamma_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{\partial \mathbf{u}}{\partial y}\cdot\mathbf{i} + \frac{\partial \mathbf{u}}{\partial x}\cdot\mathbf{j},\end{aligned} \tag{18}$$

Here, we have taken into account that the base vectors are constant with respect to differentiation. Similarly, in the local frame we have the linear strains components

$$\begin{aligned}\varepsilon_X &= \frac{\partial u_X}{\partial X} = \frac{\partial \mathbf{u}}{\partial X}\cdot\mathbf{e}_X, \\ \varepsilon_Y &= \frac{\partial u_Y}{\partial Y} = \frac{\partial \mathbf{u}}{\partial Y}\cdot\mathbf{e}_Y, \\ \gamma_{XY} &= \frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X} = \frac{\partial \mathbf{u}}{\partial Y}\cdot\mathbf{e}_X + \frac{\partial \mathbf{u}}{\partial X}\cdot\mathbf{e}_Y .\end{aligned} \tag{19}$$

or correspondingly non-linear ones

$$\begin{aligned}\varepsilon_X &= \frac{\partial \mathbf{u}}{\partial X}\cdot\mathbf{e}_X + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial X}\cdot\frac{\partial \mathbf{u}}{\partial X}, \\ \varepsilon_Y &= \frac{\partial \mathbf{u}}{\partial Y}\cdot\mathbf{e}_Y + \frac{1}{2}\frac{\partial \mathbf{u}}{\partial Y}\cdot\frac{\partial \mathbf{u}}{\partial Y}, \\ \gamma_{XY} &= \frac{\partial \mathbf{u}}{\partial Y}\cdot\mathbf{e}_X + \frac{\partial \mathbf{u}}{\partial X}\cdot\mathbf{e}_Y + \frac{\partial \mathbf{u}}{\partial X}\cdot\frac{\partial \mathbf{u}}{\partial Y}.\end{aligned} \tag{20}$$

When applying now the differentiation rules (14), and consider the result at the origin, we get

$$\begin{aligned}
\varepsilon_X &= \varepsilon_\alpha = \frac{\partial \mathbf{u}}{h_\alpha \partial \alpha} \bullet \mathbf{e}_\alpha + \frac{1}{2} \frac{\partial \mathbf{u}}{h_\alpha \partial \alpha} \bullet \frac{\partial \mathbf{u}}{h_\alpha \partial \alpha}, \\
\varepsilon_Y &= \varepsilon_\beta = \frac{\partial \mathbf{u}}{h_\beta \partial \beta} \bullet \mathbf{e}_\beta + \frac{1}{2} \frac{\partial \mathbf{u}}{h_\beta \partial \beta} \bullet \frac{\partial \mathbf{u}}{h_\beta \partial \beta}, \\
\gamma_{XY} &= \gamma_{\alpha\beta} = \frac{\partial \mathbf{u}}{h_\beta \partial \beta} \bullet \mathbf{e}_\alpha + \frac{\partial \mathbf{u}}{h_\alpha \partial \alpha} \bullet \mathbf{e}_\beta + \frac{\partial \mathbf{u}}{h_\alpha \partial \alpha} \bullet \frac{\partial \mathbf{u}}{h_\beta \partial \beta}.
\end{aligned} \tag{21}$$

The fact which is now important to notice is that the kinematics is given in curvilinear coordinates, including the unit vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ which are not constant with respect to differentiation, and their derivatives according to rule (7) have to be taken into account.

All the strain components can now be calculated very mechanically, independently of how complicated the kinematics or the structure under consideration is.

3.1 The gradient

The gradient and the divergence expressions in curvilinear coordinates are derived here as examples of the use of the method of local Cartesian frame. Further, the divergence expression is needed in the derivation of the integration by parts formula.

In Cartesian coordinates the definition the gradient of a scalar f is

$$\text{grad} f \equiv \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{22}$$

or using the local frame

$$\text{grad} f = \frac{\partial f}{\partial X}\mathbf{e}_X + \frac{\partial f}{\partial Y}\mathbf{e}_Y. \tag{23}$$

Thus, at the local origin due to (13) and (14)

$$\text{grad} f = \frac{1}{h_\alpha}\frac{\partial f}{\partial \alpha}\mathbf{e}_\alpha + \frac{1}{h_\beta}\frac{\partial f}{\partial \beta}\mathbf{e}_\beta. \tag{24}$$

This is the standard mathematics formula for the gradient in curvilinear coordinates.
In polar coordinates, due to (12), we get

$$\text{grad} f = \frac{1}{h_r}\frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{h_\theta}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta. \tag{25}$$

3.2 The Divergence

For a vector **F** we have the alternative representations

$$\begin{aligned}
\mathbf{F} &= F_x(x,y)\mathbf{i} + F_y(x,y))\mathbf{j}, \\
&= F_X(X,Y)\mathbf{e}_X + F_Y(X,Y)\mathbf{e}_Y, \\
&= F_\alpha(\alpha,\beta)\mathbf{e}_\alpha(\alpha,\beta) + F_\beta(\alpha,\beta)\mathbf{e}_\beta(\alpha,\beta).
\end{aligned} \tag{26}$$

Again it is emphasized that although point P for the origin of the X,Y-system can be anywhere, during the following steps, it and the directions of the X- and Y-axes are fixed.

In Cartesian coordinates the definition for the divergence is

$$\mathrm{div}\mathrm{F} \equiv \nabla \bullet \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = \frac{\partial F_X}{\partial X} + \frac{\partial F_Y}{\partial Y}. \tag{27}$$

Now

$$F_X = \mathbf{F} \bullet \mathbf{e}_X , \quad F_Y = \mathbf{F} \bullet \mathbf{e}_y \tag{28}$$

so

$$\frac{\partial F_X}{\partial X} = \frac{\partial}{\partial X}(\mathbf{F} \bullet \mathbf{e}_X) = \frac{\partial \mathbf{F}}{\partial X} \bullet \mathbf{e}_X , \quad \frac{\partial F_Y}{\partial Y} = \frac{\partial}{\partial Y}(\mathbf{F} \bullet \mathbf{e}) = \frac{\partial \mathbf{F}}{\partial Y} \bullet \mathbf{e}_Y \tag{29}$$

as in the local frame $\mathbf{e}_X$ and $\mathbf{e}_Y$ are constants. Thus,

$$\mathrm{div}\mathbf{F} = \frac{\partial \mathbf{F}}{\partial X} \bullet \mathbf{e}_X + \frac{\partial \mathbf{F}}{\partial Y} \bullet \mathbf{e}_Y . \tag{30}$$

This type of representation is suitable for the method of local Cartesian frame. At the local origin due to formulas (13) and (14)

$$\mathrm{div}\mathbf{F} = \frac{1}{h_\alpha}\frac{\partial \mathbf{F}}{\partial \alpha} \bullet \mathbf{e}_\alpha + \frac{1}{h_\beta}\frac{\partial \mathbf{F}}{\partial \beta} \bullet \mathbf{e}_\beta . \tag{31}$$

Employing the last form (26), we obtain

$$\frac{1}{h_\alpha}\frac{\partial \mathbf{F}}{\partial \alpha} \bullet \mathbf{e}_\alpha = \frac{1}{h_\alpha}\left[\frac{\partial}{\partial \alpha}\left(F_\alpha \mathbf{e}_\alpha + F_\beta \mathbf{e}_\beta\right)\right] \bullet \mathbf{e}_\alpha = \frac{1}{h_\alpha}\left(\frac{\partial F_\alpha}{\partial \alpha} + \frac{F_\beta}{h_\beta}\frac{\partial h_\alpha}{\partial \beta}\right). \tag{32}$$

Use have been made of expressions (7) and of the properties of the scalar product. Similarly,

$$\frac{1}{h_\beta}\frac{\partial \mathbf{F}}{\partial \beta} \bullet \mathbf{e}_\beta = \frac{1}{h_\beta}\left(\frac{\partial F_\beta}{\partial \beta} + \frac{F_\alpha}{h_\alpha}\frac{\partial h_\beta}{\partial \alpha}\right) \tag{33}$$

and thus

$$\begin{aligned}\mathrm{div}\,\mathbf{F} &= \frac{1}{h_\alpha}\frac{\partial F_\alpha}{\partial \alpha} + \frac{1}{h_\alpha h_\beta}\frac{\partial h_\alpha}{\partial \beta}F_\beta + \frac{1}{h_\beta}\partial\frac{\partial F_\beta}{\partial \beta} + \frac{1}{h_\alpha h_\beta}\frac{\partial h_\beta}{\partial \alpha}F_\alpha \\ &= \frac{1}{h_\alpha h_\beta}\left[\frac{\partial}{\partial \alpha}\left(h_\beta F_\alpha\right) + \frac{\partial}{\partial \beta}\left(h_\alpha F_\beta\right)\right],\end{aligned} \tag{34}$$

agreeing with the standard formula given in the literature. In polar coordinates (34) takes the form

$$\mathrm{div}\,\mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial \theta}(F_\theta)\right] = \frac{\partial F_r}{\partial r} + \frac{F_r}{r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}. \tag{35}$$

4. PRINCIPLE OF VIRTUAL WORK

The principle of virtual work has a central role in teaching structural mechanics. It can be employed in analytical applications but it also forms the starting point for numerical methods, especially the finite element method. It is thus a unifying principle which should be clearly understood by the student. It is well-known that the method can be used to derive equilibrium equations in a systematic way. Here, this idea is employed in connection with the method of local Cartesian frame.

The principle of virtual work can be expressed as

$$\delta W^i + \delta W^e = 0 . \tag{36}$$

Here for a two-dimensional continuum (and small strains) the virtual work of internal forces

$$\delta W^i = -\int_A \left(\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \tau_{xy} \delta \gamma_{xy} \right) \mathrm{d}A \tag{37}$$

and the virtual work of external forces, correspondingly

$$\delta W^e = \int_A \left(f_x \delta u + f_y \delta v \right) \mathrm{d}A + \int_{s_t} \left(t_x \delta u + t_y \delta v \right) \mathrm{d}s \tag{38}$$

The virtual strains are obtained by variation of the general strain-displacement relations:

5.1 One dimension

Integration by parts is an important mathematical manipulation which is needed in structural mechanics especially in connection with the principle of virtual work.
The starting point is the relation

$$\int_a^b \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x = [u]_a^b \tag{39}$$

for an arbitrary (smooth) function $u(x)$. Inserting $u = fg$, where $f(x)$ and $g(x)$ are two functions, produces with some arrangement the integration by parts formula

$$\int_a^b \frac{\mathrm{d}f}{\mathrm{d}x} g \,\mathrm{d}x = [fg]_a^b - \int_a^b f \frac{\mathrm{d}g}{\mathrm{d}x} \mathrm{d}x . \tag{40}$$

5.2 Two dimensions

The starting point is the Gauss formula which is in two dimensions

$$\int_A \nabla \bullet \mathbf{F} \mathrm{d}A = \int_s \mathbf{F} \bullet \mathbf{n} \mathrm{d}s . \tag{41}$$

The meaning of the notations is obvious. In Cartesian coordinates (41) obtains the form

$$\int_A \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} \right) \mathrm{d}x \mathrm{d}y = \int_s (F_x n_x + F_y n_y) \mathrm{d}s \tag{42}$$

and in curvilinear coordinates ($\mathrm{d}A = \mathrm{d}X\mathrm{d}Y = h_\alpha \mathrm{d}\alpha \, h_\beta \mathrm{d}\beta = h_\alpha h_\beta \mathrm{d}\alpha \, \mathrm{d}\beta$ and $\nabla \bullet \mathbf{F}$ is obtained from (31)):

$$\int_{\alpha,\beta}\left[\frac{\partial}{\partial\alpha}(h_\beta F_\alpha)+\frac{\partial}{\partial\beta}(h_\alpha F_\beta)\right]\mathrm{d}\alpha\,\mathrm{d}\beta=\int_s(F_\alpha n_\alpha+F_\beta n_\beta)\mathrm{d}s\,. \tag{43}$$

The α,β notation is used to indicate that the area integral is to be taken in the α,β-plane. Defining temporarily functions u and v by $u=h_\beta F_\alpha$, $v=h_\alpha F_\beta$ gives the form

$$\int_{\alpha,\beta}\left(\frac{\partial u}{\partial\alpha}+\frac{\partial v}{\partial\beta}\right)\mathrm{d}\alpha\,\mathrm{d}\beta=\int_s\left(\frac{n_\alpha}{h_\beta}u+\frac{n_\beta}{h_\alpha}v\right)\mathrm{d}s\,. \tag{44}$$

Finally making the consecutive selections, $u=fg$ and $v=0$, $u=0$, where $f(\alpha,\beta)$ and $g(\alpha,\beta)$ are two functions, we arrive at the integration by parts formulas

$$\begin{aligned}\int_{\alpha,\beta}f\frac{\partial g}{\partial\alpha}\mathrm{d}\alpha\,\mathrm{d}\beta&=\int_s\frac{n_\alpha}{h_\beta}fg\,\mathrm{d}s-\int_{\alpha,\beta}\frac{\partial f}{\partial\alpha}g\,\mathrm{d}\alpha\,\mathrm{d}\beta\,,\\ \int_{\alpha,\beta}f\frac{\partial g}{\partial\beta}\mathrm{d}\alpha\,\mathrm{d}\beta&=\int_s\frac{n_\beta}{h_\alpha}fg\,\mathrm{d}s-\int_{\alpha,\beta}\frac{\partial f}{\partial\beta}g\,\mathrm{d}\alpha\,\mathrm{d}\beta\,.\end{aligned} \tag{45}$$

In polar coordinates $\mathrm{d}A=\mathrm{d}X\,\mathrm{d}Y=\mathrm{d}r\,r\,\mathrm{d}\theta=r\,\mathrm{d}r\,\mathrm{d}\theta$ and equations (46) obtain the forms

$$\begin{aligned}\int_{r,\theta}f\frac{\partial g}{\partial r}\mathrm{d}r\,\mathrm{d}\theta&=\int_s\frac{n_r}{r}fg\,\mathrm{d}s-\int_{r,\theta}\frac{\partial f}{\partial r}g\,\mathrm{d}r\,\mathrm{d}\theta\,,\\ \int_{r,\theta}f\frac{\partial g}{\partial\theta}\mathrm{d}r\,\mathrm{d}\theta&=\int_s n_\theta fg\,\mathrm{d}s-\int_{r,\theta}\frac{\partial f}{\partial\theta}g\,\mathrm{d}r\,\mathrm{d}\theta\,.\end{aligned} \tag{46}$$

5. CONCLUDING COMMENTS

Above, there are given all the mathematical tools needed to allow the application of the method of local Cartesian frame. Although most of the relevant formulas have probably been presented earlier to the students in some mathematics courses, it is certainly a good idea to go through these details again when teaching structural mechanics. As seen above, this does not demand too much effort. In our opinion this effort is more than paid back when applications with curved structures are encountered. The need to employ complicated strain deduction figures and free body diagrams disappears.

REFERENCES

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