HOW TO WORK WITH CURVED STRUCTURES; APPLICATIONS

J. PAAVOLA and E.-M. SALONEN
Department of Civil and Structural Engineering, Aalto University,
P.O.Box 12100, FI-00076, Aalto, Finland
e-mail: Juha.Paavola@aalto.fi

EXTENDED ABSTRACT

This is a companion paper to the one titled: How to work with curved structures; theory. The general theory is presented in the cited paper. In the present paper two specific example applications are given in detail. Provided that the main ingredients in the method of local Cartesian frame for curved structures are explained in the theory paper, we here just shortly describe the two applications.

The first application concerns the analysis of a circular disk. Polar coordinates are employed. The conventional procedure applied in textbooks in this case is again based on carefully drawn figures showing the initial and deformed geometry for a small material element. Deducing the relevant expressions from the figures demands rather careful interpretations. Our approach is based on the use of a local Cartesian frame. Further, after the strains have been determined, the local equilibrium equations and the traction boundary conditions are arrived at by employing the principle of virtual work. Integration by parts in two dimensions is needed in the manipulations. This part of mathematics may be somewhat unfamiliar to the students. However, in introducing the most important principle of virtual work in general, integration by parts must be mastered, so this should not be a grave problem.

The second application concerns the analysis of a circular beam in two loading cases. Emphasis is placed on the importance of the corresponding kinematic assumptions. The curvilinear coordinates are now the beam axis arc length and two rectangular axes perpendicular to the beam axis. To determine the strains correctly from figures describing the geometry in the original and in the deformed state seems to us as a nearly impossible task. The method of local Cartesian frame works easily. In the equilibrium equations derivation, which is based on the principle of virtual work, integration by parts is needed only in one dimension. This tool should be already rather familiar to the students.

The meaning of the papers considered, is not just to derive the basic equations of classical mechanics, but to derive them in a systematic way students can easier assimilate. According to the feedback of students, it is obvious that even the complicated equations of the shell theory have got a novel role, when the background of each term will get a clear physical meaning.

Other concepts based on various kinematical assumptions, such as sectorial coordinate with thin-walled structures, may be derived simply as well.

KEYWORDS

Teaching Techniques for Mechanics, Curved Structures, Circular disk, Circular beam

1. INTRODUCTION

This is a companion paper to the paper of this conference titled: How to work with curved structures; theory. This will be referred to hereon as the theory paper. The general theory is presented in the cited paper. In the present paper, two specific example applications are given in detail.

The first application concerns the analysis of a circular disk. Polar coordinates are employed. The second application concerns the analysis of a circular beam. The curvilinear coordinates are now the beam arc length parameter and two rectangular axes perpendicular to the beam axis.

We repeat from the theory paper the basic relations used in the method of local Cartesian frame:

$$\mathbf{e}_X = \mathbf{e}_\alpha , \quad \mathbf{e}_Y = \mathbf{e}_\beta \tag{1}$$

and

$$\frac{\partial}{\partial X} = \frac{1}{h_\alpha}\frac{\partial}{\partial \alpha} , \quad \frac{\partial}{\partial Y} = \frac{1}{h_\beta}\frac{\partial}{\partial \beta} . \tag{2}$$

The meaning of the notations are shown in Figure 1, and explained in the theory paper.

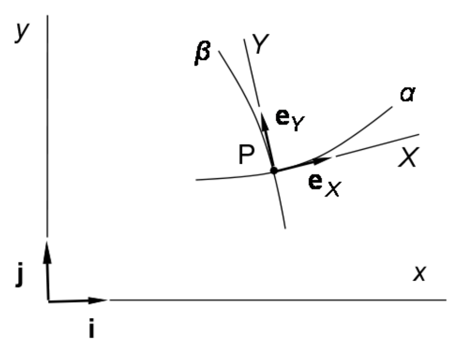

Figure 1: The local frame.

Additionally, we may need the derivatives of the unit vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_\beta$ with respect to α and β. In connection with the principle of virtual work we will need also the integration by parts formulae. These both are derived in the theory paper. However, in what follows we will not represent general formulas for strains and general equilibrium equations, (Paavola and Salonen (2004)). Instead, in the two applications we will derive the necessary relations directly and not via general formulae.

2. CIRCULAR DISK
2.1 Polar coordinates

We consider a circular disk with a radius R (Figure 2). Polar coordinates r and θ are employed in the analysis. No dependence of the relevant quantities in the perpendicular direction to the disk is assumed.

We repeat here from the theory paper the relevant formulae in polar coordinates. The scale factors are $h_r = 1$, $h_\theta = r$. The counterparts of (1) and (2) are

$$\mathbf{e}_X = \mathbf{e}_r\,, \quad \mathbf{e}_Y = \mathbf{e}_\theta \tag{3}$$

and

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial r}\,, \quad \frac{\partial}{\partial Y} = \frac{1}{r}\frac{\partial}{\partial \theta}\,. \tag{4}$$

Further, the derivatives of the unit vectors are

$$\frac{\partial \mathbf{e}_r}{\partial r} = \mathbf{0}\,, \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta\,, \quad \frac{\partial \mathbf{e}_\theta}{\partial r} = \mathbf{0}\,, \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r\,. \tag{5}$$

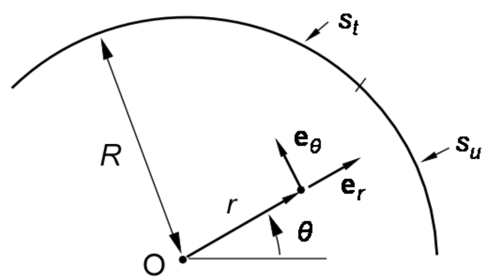

Figure 2: Part of a circular disk and notation.

2.2 Strains

The displacement field ***u*** has the alternative representations

$$\begin{aligned}\mathbf{u} &= u_X \mathbf{e}_X + u_Y \mathbf{e}_Y\,,\\ &= u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta\,.\end{aligned} \tag{6}$$

In the theory paper, the small strain expressions are derived in the local frame:

$$\begin{aligned}\varepsilon_X &= \frac{\partial u_X}{\partial X} = \frac{\partial \mathbf{u}}{\partial X} \bullet \mathbf{e}_X\,,\\ \varepsilon_Y &= \frac{\partial u_Y}{\partial Y} = \frac{\partial \mathbf{u}}{\partial Y} \bullet \mathbf{e}_Y\,,\\ \gamma_{XY} &= \frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X} = \frac{\partial \mathbf{u}}{\partial Y} \bullet \mathbf{e}_X + \frac{\partial \mathbf{u}}{\partial X} \bullet \mathbf{e}_Y\,.\end{aligned} \tag{7}$$

Thus, applying (3) and (4), at the local origin, the strain components are resolved:

$$\begin{aligned}\varepsilon_r &= \varepsilon_X = \frac{\partial \mathbf{u}}{\partial r} \bullet \mathbf{e}_r\,,\\ \varepsilon_\theta &= \varepsilon_Y = \frac{1}{r}\frac{\partial \mathbf{u}}{\partial \theta} \bullet \mathbf{e}_\theta\,,\\ \gamma_{r\theta} &= \gamma_{XY} = \frac{1}{r}\frac{\partial \mathbf{u}}{\partial \theta} \bullet \mathbf{e}_r + \frac{\partial \mathbf{u}}{\partial r} \bullet \mathbf{e}_\theta\,.\end{aligned} \tag{8}$$

The last form (6) is substituted in (8) and formulae (5) are used. We obtain in detail

$$\varepsilon_r = \left(\frac{\partial u_r}{\partial r}\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\right) \bullet \mathbf{e}_r = \frac{\partial u_r}{\partial r}\,,$$

$$\varepsilon_\theta = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\mathbf{e}_r + u_r\mathbf{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta - u_\theta\mathbf{e}_r\right)\bullet\mathbf{e}_\theta = \frac{1}{r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)\ , \tag{9}$$

$$\gamma_{r\theta} = \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}\mathbf{e}_r + u_r\mathbf{e}_\theta + \frac{\partial u_\theta}{\partial \theta}\mathbf{e}_\theta - u_\theta\mathbf{e}_r\right)\bullet\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial r}\mathbf{e}_r + \frac{\partial u_\theta}{\partial r}\mathbf{e}_\theta\right)\bullet\mathbf{e}_\theta$$
$$= \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}.$$

Based on classical approaches (Timoshenko and Goodier (1951, p. 65-66)), these results are obtained from a rather awkward differential geometry figure.

2.3 Equilibrium

The well-known general form of stress equilibrium equations for a continuum is $\operatorname{div}\boldsymbol{\sigma} + \mathbf{f} = \mathbf{0}$, where $\boldsymbol{\sigma}$ is the stress tensor and $\mathbf{f}$ the body force vector intensity (per volume). We could continue by using the method of local Cartesian frame and dyadic representation. A rather long manipulation is needed to give the final equilibrium equations. They are not given here. Further, the equation referred to is clearly not a suitable starting point for basic courses. The equation itself is probably not familiar and the manipulations needed are rather tedious even when the present polar coordinate case is considered. However, the principle of virtual work gives an alternative way to produce the equilibrium equations.

The principle of virtual work is applied for the equilibrium consideration, and it can be expressed as

$$\delta W^i + \delta W^e = 0\,. \tag{10}$$

Here for a two-dimensional continuum (assuming small strains) the virtual work of internal forces is expressed by:

$$\delta W^i = -\int_A \left(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy}\right)\mathrm{d}A \tag{11}$$

and the virtual work of external forces, correspondingly is,

$$\delta W^e = \int_A \left(f_x\delta u + f_y\delta v\right)\mathrm{d}A + \int_{s_t}\left(t_x\delta u + t_y\delta v\right)\mathrm{d}s \tag{12}$$

The virtual strains are obtained by variation of the general strain-displacement relations. Boundary s_t is that part of the total boundary where the traction $\mathbf{t}$ is given. Correspondingly, in Figure 2 the notation s_u refers to that part of the total boundary where the displacement $\mathbf{u}$ is given.

The integrands in (11) and (12) are because of their physical meaning, scalar quantities and therefore invariant with respect to coordinate transformations. With reference to polar coordinates, the following may be written:

$$\begin{aligned}
&\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\varepsilon_{xy} = \sigma_X\delta\varepsilon_X + \sigma_Y\delta\varepsilon_Y + \tau_{XY}\delta\gamma_{XY}\\
&= \sigma_r\delta\varepsilon_r + \sigma_\theta\delta\varepsilon_\theta + \tau_{r\theta}\delta\gamma_{r\theta}\ ,\\
&f_x\delta u + f_y\delta v = f_X\delta u_X + f_Y\delta u_Y = f_r\delta u_r + f_\theta\delta u_\theta\ ,\\
&t_x\delta u + t_y\delta v = t_X\delta u_X + t_Y\delta u_Y = t_r\delta u_r + t_\theta\delta u_\theta\ .
\end{aligned} \tag{13}$$

Thus, the principle of virtual work obtains in polar coordinates the form

$$
\begin{aligned}
&-\int_A \left(\sigma_r \delta\varepsilon_r + \sigma_\theta \delta\varepsilon_\theta + \tau_{r\theta}\delta\gamma_{r\theta}\right)\mathrm{d}A \\
&+\int_A \left(f_r \delta u_r + f_\theta \delta u_\theta\right)\mathrm{d}A + \int_{s_t}\left(t_r \delta u_r + t_\theta \delta u_\theta\right)\mathrm{d}s = 0\,,
\end{aligned}
\tag{14}
$$

where the virtual strain components are expressed from (9)

$$
\delta\varepsilon_r = \frac{\partial \delta u_r}{\partial r}\,,\quad \delta\varepsilon_\theta = \frac{1}{r}\left(\delta u_r + \frac{\partial \delta u_r}{\partial \theta}\right),\quad \delta\gamma_{r\theta} = \frac{1}{r}\left(\frac{\partial \delta u_r}{\partial \theta} - \delta u_\theta\right) + \frac{\partial \delta u_\theta}{\partial r}\,. \tag{15}
$$

Upon substitution $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ in Eqn. (14), the following simplifications are obtained:

first the form

$$
\begin{aligned}
&-\int_{r,\theta}\left[r\sigma_r \frac{\partial \delta u_r}{\partial r} + \sigma_\theta\left(\delta u_r + \frac{\partial \delta u_\theta}{\partial \theta}\right) + \tau_{r\theta}\left(\frac{\partial \delta u_r}{\partial \theta} - \delta u_\theta\right) + r\tau_{r\theta}\frac{\partial \delta u_\theta}{\partial r}\right]\mathrm{d}r\mathrm{d}\theta \\
&\quad + \int_{r,\theta}\left(r f_r \delta u_r + r f_\theta \delta u_\theta\right)\mathrm{d}r\mathrm{d}\theta + \int_{s_t}\left(t_r \delta u_r + t_\theta \delta u_\theta\right)\mathrm{d}s = 0\,.
\end{aligned}
\tag{16}
$$

Then, using integration by parts (with the formulae derived in the theory paper):

$$
\begin{aligned}
\int_{\mathrm{r},\theta} f\frac{\partial g}{\partial r}\mathrm{d}r\,\mathrm{d}\theta &= \int_s \frac{n_r}{r} f g\,\mathrm{d}s - \int_{\mathrm{r},\theta}\frac{\partial f}{\partial r} g\,\mathrm{d}r\,\mathrm{d}\theta\,,\\
\int_{\mathrm{r},\theta} f\frac{\partial g}{\partial \theta}\mathrm{d}r\,\mathrm{d}\theta &= \int_s n_\theta f g\,\mathrm{d}s - \int_{\mathrm{r},\theta}\frac{\partial f}{\partial \theta} g\,\mathrm{d}r\,\mathrm{d}\theta\,.
\end{aligned}
\tag{17}
$$

These are applied to eliminate the derivatives on the virtual displacement components δu_r and u_θ. The result is

$$
\begin{aligned}
&\int_{r,\theta}\left\{\left[\frac{\partial(r\sigma_r)}{\partial r} - \sigma_\theta + \frac{\partial \tau_{r\theta}}{\partial \theta} + r f_r\right]\delta u_r + \left[\frac{\partial \sigma_\theta}{\partial \theta} + \tau_{r\theta} + \frac{\partial(r\tau_{r\theta})}{\partial r} + r f_\theta\right]\delta u_\theta\right\}\mathrm{d}r\mathrm{d}\theta \\
&\quad + \int_{s_t}\left\{\left[t_r - n_r\sigma_r - n_\theta\tau_{r\theta}\right]\delta u_r + \left[t_\theta - n_\theta\sigma_\theta - n_r\tau_{r\theta}\right]\delta u_\theta\right\}\mathrm{d}s = 0
\end{aligned}
\tag{18}
$$

It should be noted that the virtual displacement components are set to vanish on s_u, which explains why the line integral is only over s_t. The equilibrium equations are thus — after some minor development:

$$
\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0\,,\\
&\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0\,,
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
t_r &= n_r\sigma_r + n_\theta\tau_{r\theta}\,,\\
t_\theta &= n_\theta\sigma_\theta + n_r\tau_{r\theta}\,.
\end{aligned}
\tag{20}
$$

The traction boundary conditions simplify with the geometry of Figure 2 to

$$
t_r = \sigma_r,\; t_\theta = \tau_{r\theta}\,. \tag{21}
$$

as on the boundary $n_r = 1$ and $n_\theta = 0$.

Classically the equilibrium equations are obtained from a free-body diagram, e.g. Timoshenko and Goodier (1951, p.55-56). Our way of derivation cannot be considered particularly short. However, the steps used contain no arbitrariness. Further, similar manipulations must be performed in numerous applications of the principle of virtual work and this example case is a good demonstration exercise for the students.

3. CIRCULAR BEAM

3.1 Coordinate system

A circular plane beam is considered using the notation of Figure 3. The beam is assumed to be symmetrical in geometry and in material properties with respect to the xy-plane. The beam is clamped at $s=0$. As seen from the figure, here the y notation is used with two meanings; as a global coordinate and also locally at the beam cross-section, but this should not cause any confusion. Further, a local coordinate z (not shown in the figure) perpendicular to the xy-plane is needed. Compared to the theory part, where only the two dimensional case was treated, we here have one additional dimension. Additional notations γ and $\mathbf{e}_Z$ with obvious meanings are introduced. The curved beam axis is taken to be an α-coordinate line; here α is associated with the arc length s. The β- and γ-coordinate lines are straight and $\beta = y$ and $\gamma = z$. The local unit vectors $\mathbf{e}_s$, $\mathbf{e}_y$ and $\mathbf{e}_z$ form a right-handed triad.

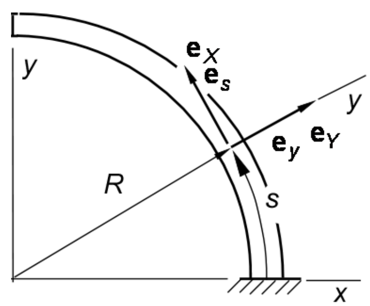

Figure 3: Circular beam.

The position vector of the beam axis is given by $\mathbf{r}_0 = R\mathbf{e}_y$ and that of the generic point P by

$$\mathbf{r}(s,y,z) = \mathbf{r}_0 + y\mathbf{e}_y + z\mathbf{e}_z = R\mathbf{e}_y + y\mathbf{e}_y + z\mathbf{e}_z = (R+y)\mathbf{e}_y + z\mathbf{e}_z . \tag{22}$$

The dependence on s comes through $\mathbf{e}_y$ which is not constant. From curve theory $\mathrm{d}\mathbf{r}_0/\mathrm{d}s = \mathbf{e}_s$ and by Frenet formulae,

$$\frac{\mathrm{d}\mathbf{e}_s}{\mathrm{d}s} = -\frac{1}{R}\mathbf{e}_y , \quad \frac{\mathrm{d}\mathbf{e}_y}{\mathrm{d}s} = \frac{1}{R}\mathbf{e}_s . \tag{23}$$

Differentiation of (22) gives

$$\begin{aligned} &\frac{\partial \mathbf{r}}{\partial s} = (R+y)\frac{\mathrm{d}\mathbf{e}_y}{\mathrm{d}s} = (R+y)\frac{1}{R}\mathbf{e}_s = \left(1+\frac{y}{R}\right)\mathbf{e}_s , \\ &\frac{\partial \mathbf{r}}{\partial y} = \mathbf{e}_y , \quad \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z . \end{aligned} \tag{24}$$

The scale factors are thus seen to be

$$h_s = \left|\frac{\partial \mathbf{r}}{\partial s}\right| = 1 + \frac{y}{R}, \quad h_y = \left|\frac{\partial \mathbf{r}}{\partial y}\right| = 1, \quad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1. \tag{25}$$

Consequently, the counterparts of (1) and (2) are now

$$\mathbf{e}_X = \mathbf{e}_s, \quad \mathbf{e}_Y = \mathbf{e}_y, \quad \mathbf{e}_Z = \mathbf{e}_z \tag{26}$$

and

$$\frac{\partial}{\partial X} = \left(1 + \frac{y}{R}\right)^{-1} \frac{\partial}{\partial s}, \quad \frac{\partial}{\partial Y} = \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial Z} = \frac{\partial}{\partial z}. \tag{27}$$

3.2 Displacement; general

We take here the Timoshenko type kinematic small displacement assumption where the beam material cross sections are assumed to move as rigid plates with no deformation in plate planes. The generic displacement vector of a generic point P becomes then

$$\mathbf{u}(s,y,z) = \left(u - y\theta_z - z\theta_y\right)\mathbf{e}_s + \left(v - z\theta_s\right)\mathbf{e}_y + \left(w + y\theta_s\right)\mathbf{e}_z. \tag{28}$$

Quantities u, v and w are the displacement components of the origin 0 of the cross-section and θ_s, θ_y and θ_z are the components of the cross-sectional rotation vector. Following certain notational convention, the component θ_y is defined as positive in the negative local y-axis direction.

3.3 Displacement; first case

To simplify the presentation we will consider here just two special cases. In the first case the loading consists of a point load P acting at the tip of beam and directed in the local positive y-axis direction. In the second case the tip load P acts in the local positive $z-$axis direction.

In the first case, due to the assumed symmetry, the displacement of point 0 must be in the xy-plane and the rotation vector must be perpendicular to the xy-plane. Thus, the only non-zero displacement components are u, v and the only non-zero rotation component is θ_z. Expression (28) simplifies to

$$\mathbf{u}(s,y) = (u - y\theta_z)\mathbf{e}_s + v\mathbf{e}_y. \tag{29}$$

3.4 Strains; first case

The relevant strain components are

$$\begin{aligned} \varepsilon_s &= \varepsilon_X = \frac{\partial \mathbf{u}}{\partial X} \bullet \mathbf{e}_X = \left(1 + \frac{y}{R}\right)^{-1} \frac{\partial \mathbf{u}}{\partial s} \bullet \mathbf{e}_s, \\ \gamma_{sy} &= \gamma_{XY} = \frac{\partial \mathbf{u}}{\partial Y} \bullet \mathbf{e}_X + \frac{\partial \mathbf{u}}{\partial X} \bullet \mathbf{e}_Y = \frac{\partial \mathbf{u}}{\partial y} \bullet \mathbf{e}_s + \left(1 + \frac{y}{R}\right)^{-1} \frac{\partial \mathbf{u}}{\partial s} \bullet \mathbf{e}_y. \end{aligned} \tag{30}$$

The derivatives

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial s} &= \left(\frac{\mathrm{d}u}{\mathrm{d}s} - y\frac{\mathrm{d}\theta_z}{\mathrm{d}s}\right)\mathbf{e}_s + (u - y\theta_z)\left(-\frac{\mathbf{e}_y}{R}\right) + \frac{\mathrm{d}v}{\mathrm{d}s}\mathbf{e}_y + v\frac{\mathbf{e}_s}{R}, \\ \frac{\partial \mathbf{u}}{\partial y} &= -\theta_z\,\mathbf{e}_s \end{aligned} \tag{31}$$

and substitution into (30) gives

$$\begin{aligned}
\varepsilon_s &= \left(1+\frac{y}{R}\right)^{-1}\left(\frac{\mathrm{d}u}{\mathrm{d}s} - y\frac{\mathrm{d}\theta_z}{\mathrm{d}s} + \frac{v}{R}\right), \\
\gamma_{sy} &= \left(1+\frac{y}{R}\right)^{-1}\left(-\frac{u-y\theta_z}{R} + \frac{\mathrm{d}v}{\mathrm{d}s}\right) - \theta_z .
\end{aligned} \tag{32}$$

To derive these exact results alternatively by some — however carefully drawn — figures is in our opinion practically impossible.

3.5 Equilibrium; first case

Again, the equilibrium equations are derived here by employing the principle of virtual work. The usual assumptions concerning stresses for plane beams is that only the components σ_s and τ_{sy} are non-zero. Thus, the virtual work of internal forces becomes

$$\delta w^{I} = \int_V \left(\sigma_s \delta\varepsilon_s + \tau_{sy}\delta\gamma_{sy}\right)\mathrm{d}V , \tag{33}$$

where the integration is over the volume of the beam. The volume element $\mathrm{d}V$ can be expressed here as

$$\mathrm{d}V = \mathrm{d}A\mathrm{d}X = \mathrm{d}Ah_s\mathrm{d}s = \mathrm{d}A\left(1+\frac{y}{R}\right)\mathrm{d}s , \tag{34}$$

where $\mathrm{d}A$ is beam cross-sectional area element. The virtual strain expressions are obtained by variations of Eqn. (32) and the virtual work expression becomes

$$\begin{aligned}
\delta w^{I} &= -\int_s \left\{ \int_A [\sigma_s \left(1+\frac{y}{R}\right)^{-1}\left(\frac{\mathrm{d}\delta u}{\mathrm{d}s} - y\frac{\mathrm{d}\delta\theta_z}{\mathrm{d}s} + \frac{\delta v}{R}\right) \right. \\
&\left. +\tau_{sy}\left(1+\frac{y}{R}\right)^{-1}\left(\frac{\mathrm{d}\delta v}{\mathrm{d}s} - \frac{\delta u}{R} - \delta\theta_z\right)\Bigg]\left(1+\frac{y}{R}\right)\mathrm{d}A \right\}\mathrm{d}s \\
&= -\int_s \left\{ \int_A \left[\sigma_s\left(\frac{\mathrm{d}\delta u}{\mathrm{d}s} - y\frac{\mathrm{d}\delta\theta_z}{\mathrm{d}s} + \frac{\delta v}{R}\right) + \tau_{sy}\left(\frac{\mathrm{d}\delta v}{\mathrm{d}s} - \frac{\delta u}{R} - \delta\theta_z\right)\right]\mathrm{d}A \right\}\mathrm{d}s .
\end{aligned} \tag{35}$$

The inner integral is over the beam cross-section and the outer over the beam axis length.

In the conventional manner, the stress resultants consisting of the normal force, the shearing force and the bending moment are defined respectively by

$$N = \int_A \sigma_s \mathrm{d}A, \quad Q_y = \int_A \tau_{sy}\mathrm{d}A, \quad M_z = \int_A \sigma_s y\,\mathrm{d}A. \tag{36}$$

The virtual work of internal forces (35) becomes then

$$\delta w^{I} = -\int_s \left[N\left(\frac{\mathrm{d}\delta u}{\mathrm{d}s} + \frac{\delta v}{R}\right) - M_z\frac{\mathrm{d}\delta\theta_z}{\mathrm{d}s} + Q_y\left(\frac{\mathrm{d}\delta v}{\mathrm{d}s} - \frac{\delta u}{R} - \delta\theta_z\right)\right]\mathrm{d}s . \tag{37}$$

The virtual work of external forces is here just the virtual work of the vertical load:

$$\delta W^{e} = P\delta v\big|_{s=l} , \tag{38}$$

where $l = \pi R/2$. The virtual work equation becomes thus

$$-\int_s \left[N\left(\frac{\mathrm{d}\delta u}{\mathrm{d}s}+\frac{\delta v}{R}\right) - M_z \frac{\mathrm{d}\delta\theta_z}{\mathrm{d}s} + Q_y \left(\frac{\mathrm{d}\delta v}{\mathrm{d}s} - \frac{\delta u}{R} - \delta\theta_z\right)\right]\mathrm{d}s + P\,\delta v\big|_{s=l} = 0 . \tag{39}$$

To deduce the equilibrium equations, the derivatives on the virtual displacement quantities must be removed by integration by parts. The corresponding formula from the theory paper becomes with x replaced by s here

$$\int_0^l \frac{\mathrm{d}f}{\mathrm{d}s} g\,\mathrm{d}s = [fg]_0^l - \int_0^l f \frac{\mathrm{d}g}{\mathrm{d}s}\mathrm{d}s . \tag{40}$$

Equation (39) is found to transform to (note that due to the clamped beam end $\delta u = \delta v = \delta\theta_z = 0$ at $s = 0$)

$$\int_s \left\{ \left[\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{Q_y}{R}\right]\delta u + \left[\frac{\mathrm{d}Q_y}{\mathrm{d}s} - \frac{N}{R}\right]\delta v - \left[\frac{\mathrm{d}M_z}{\mathrm{d}s} - Q_y\right]\delta\theta \right\}\mathrm{d}s$$
$$+\left[-N\delta u + \left(-Q_y + P\right)\delta v + M_z\delta\theta\right]\Big|_{s=l} = 0 . \tag{41}$$

Thus, the field equilibrium equations are

$$\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{Q_y}{R} = 0, \quad \frac{\mathrm{d}Q_y}{\mathrm{d}s} - \frac{N}{R} = 0, \quad \frac{\mathrm{d}M_z}{\mathrm{d}s} - Q_y = 0 \tag{42}$$

and the traction boundary conditions at $s = l$ are

$$N = 0, \ Q_y = P, \quad M_z = 0 . \tag{43}$$

In this one-dimensional case the free body diagram approach can produce these relations relatively easily. The virtual work approach presented here may be considered as an alternative method to direct establishment of equilibrium equations, once again demonstrating to the students the importance of the principle of the virtual work.

3.6 Displacement; second case

Now due to the assumed symmetry, the displacement of point 0 must perpendicular to the xy-plane and the rotation vector must be in xy-plane. Thus, the only non-zero displacement component is w and the only non-zero rotation components are θ_s and θ_y. Expression (28) simplifies to

$$\mathbf{u}(s,y,z) = -z\theta_y\mathbf{e}_s - z\theta_s\mathbf{e}_y + (w + y\theta_s)\mathbf{e}_z . \tag{44}$$

3.7 Strains; second case

Due to the page limitations, the following presentation is outlined briefly. However, the steps needed are completely similar to those used in the first loading case. The relevant strain components are now

$$\begin{aligned}
\varepsilon_s &= \varepsilon_X = \frac{\partial\mathbf{u}}{\partial X}\bullet\mathbf{e}_X = \left(1+\frac{y}{R}\right)^{-1}\frac{\partial\mathbf{u}}{\partial s}\bullet\mathbf{e}_s ,\\
\gamma_{sy} &= \gamma_{XY} = \frac{\partial\mathbf{u}}{\partial Y}\bullet\mathbf{e}_X + \frac{\partial\mathbf{u}}{\partial X}\bullet\mathbf{e}_Y = \frac{\partial\mathbf{u}}{\partial y}\bullet\mathbf{e}_s + \left(1+\frac{y}{R}\right)^{-1}\frac{\partial\mathbf{u}}{\partial s}\bullet\mathbf{e}_y ,\\
\gamma_{sz} &= \gamma_{XZ} = \frac{\partial\mathbf{u}}{\partial Z}\bullet\mathbf{e}_X + \frac{\partial\mathbf{u}}{\partial X}\bullet\mathbf{e}_Z = \frac{\partial\mathbf{u}}{\partial z}\bullet\mathbf{e}_s + \left(1+\frac{y}{R}\right)^{-1}\frac{\partial\mathbf{u}}{\partial s}\bullet\mathbf{e}_z .
\end{aligned} \tag{45}$$

By differentiating the displacement expression (44) with respect to s, y and z and substituting the results in (45) the following are obtained:

$$\begin{aligned}
\varepsilon_s &= -\left(1+\frac{y}{R}\right)^{-1} z\left(\frac{\mathrm{d}\theta_y}{\mathrm{d}s}+\frac{\theta_s}{R}\right),\\
\gamma_{sy} &= -\left(1+\frac{y}{R}\right)^{-1} z\left(\frac{\mathrm{d}\theta_s}{\mathrm{d}s}-\frac{\theta_y}{R}\right),\\
\gamma_{sz} &= \left(1+\frac{y}{R}\right)^{-1}\left(\frac{\mathrm{d}w}{\mathrm{d}s}+y\frac{\mathrm{d}\theta_s}{\mathrm{d}s}\right)-\theta_y\ .
\end{aligned} \tag{46}$$

3.8 Equilibrium; second case

The virtual work of internal forces

$$\delta w^I = \int_V \left(\sigma_s\delta\varepsilon_s + \tau_{sy}\delta\gamma_{sy} + \tau_{sz}\delta\gamma_{sz}\right)\mathrm{d}V \tag{47}$$

becomes finally

$$\delta w^I = -\int_s \left[M_s\left(\frac{\mathrm{d}\delta\theta_s}{\mathrm{d}s}-\frac{\delta\theta_y}{R}\right)-M_y\left(\frac{\mathrm{d}\delta\theta_y}{\mathrm{d}s}+\frac{\delta\theta_s}{R}\right)+Q_z\left(\frac{\mathrm{d}\delta w}{\mathrm{d}s}-\delta\theta_y\right)\right]\mathrm{d}s \tag{48}$$

where the stress resultants are

$$M_s = \int_A \left(y\tau_{sz} - z\tau_{sy}\right)\mathrm{d}A,\quad M_y = \int_A \sigma_s z\,\mathrm{d}A,\quad Q_z = \int_A \tau_{sz}\,\mathrm{d}A. \tag{49}$$

The external virtual work from the transverse load P at the beam tip is

$$\delta W^e = P\,\delta w\big|_{s=l} \tag{50}$$

The virtual work equation gives after the necessary integration by parts manipulations the field equations

$$\frac{\mathrm{d}M_s}{\mathrm{d}s}+\frac{M_y}{R}=0,\quad \frac{\mathrm{d}M_y}{\mathrm{d}s}-\frac{M_s}{R}-Q_z=0,\quad \frac{\mathrm{d}Q_z}{\mathrm{d}s}=0 \tag{51}$$

and the traction boundary conditions

$$M_s = 0,\quad M_y = 0,\quad Q_z = P \tag{52}$$

at $s = l$.

4. CONCLUSIONS

The present paper shows some simple examples to demonstrate the use of the method of local Cartesian frame by emphasizing the role of the kinematics to take into account the loading and the geometry of the structure considered.

REFERENCES

1. Paavola, J. and E.-M. Salonen (2004 a), Coping with curvilinear coordinates in solid mechanics, *International Journal of Mechanical Engineering Education*, **32:1**, 1-10.
2. Timoshenko, S and J. N.Goodier (1951), Theory of Elasticity, 2nd edn, McGraw-Hill, New York.